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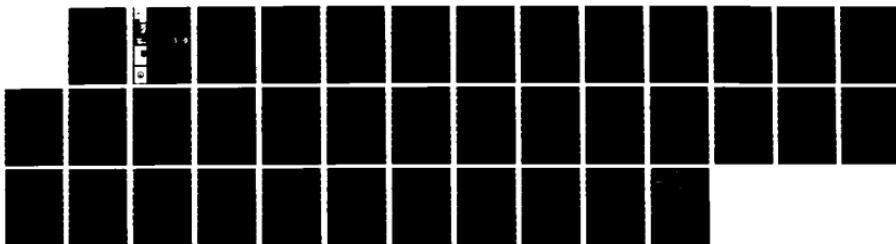
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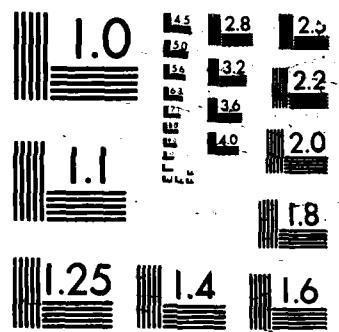
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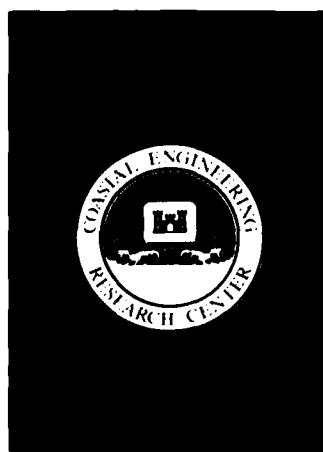
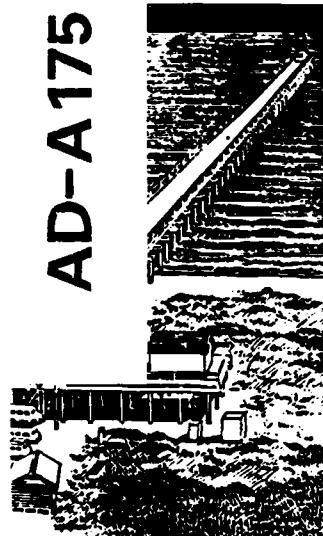






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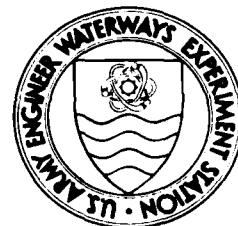
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# PARABOLIC WAVE EQUATION FOR SURFACE WATER WAVES

by

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November 1986

Final Report

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19 ABSTRACT (Continue on reverse if necessary and identify by block number)  In this report, the mild-slope equation describing the combined refraction and diffraction of small amplitude water waves is re-derived and reviewed. The relationship between the mild-slope equation and the ray theory is presented. The physical justifications for the parabolic approximation are then discussed. Parabolic wave equations in various forms are derived to investigate the effects of energy dissipation, reflection, and nonlinearity.												
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## PREFACE

This work was authorized as part of the Civil Works Research and Development Program of the Office, Chief of Engineers (OCE), US Army. The research work unit which funded this work, Regional Coastal Processes Numerical Modeling System, is part of the Shore Protection and Restoration Program. Mr. J. H. Lockhart, Jr., was the OCE Technical Monitor during preparation and publication of this report.

This report was prepared by Dr. Philip L.-F. Liu, a professor and Associate Dean of Engineering College at Cornell University, for the US Army Engineer Waterways Experiment Station (WES). Work was conducted under the direction of Dr. J. R. Houston, Chief, Coastal Engineering Research Center (CERC), WES; Mr. C. C. Calhoun, Jr., Assistant Chief, CERC; and Mr. H. L. Butler, Chief, Coastal Processes Branch.

Commander and Director of WES during completion of this effort was COL Dwayne G. Lee, CE. Dr. Robert W. Whalin was Technical Director.



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## 1. INTRODUCTION

The parabolic equation method was first introduced by Leontovich and Fock (1944) in their studies of tropospheric radio wave propagation over a long distance. They were concerned with calculating the diffraction of radio waves by the spherical shape of the earth. The predominate direction of the wave propagation, which is needed to make the small-angle parabolic approximation, was the line of sight between the antenna and the horizon. This method was later applied to many other radio wave diffraction problems such as high frequency scattering by obstacles of various shapes.

Applications of the parabolic equation method were quickly extended to wave propagation problems in other fields of physical sciences, such as nonlinear optics (Svelto, 1974), plasma physics (Karpman, 1975), geophysics (Claerbout, 1976), and underwater acoustics (Tappert, 1977). Efficient numerical algorithms have also been developed to solve parabolic equations, i.e., linear and nonlinear Schrödinger equations, for large scale problems.

Adoption of the parabolic equation method in computations of linear surface water wave propagation started in the late-70's. Based on the parabolic approximation, Liu and Mei (1976) presented an analytical solution for the wave field in the neighborhood of a breakwater located on an uniform sloping beach. Rigorous derivations of the parabolic wave equations describing the general combined refraction and diffraction were given by Radder (1979) and Lozano and Liu (1980). Lozano and Liu employed the multiple scales perturbation method and established the relationship between ray theory and the parabolic method equation. On the other hand, using an operator splitting technique (Corones, 1975), Radder demonstrated that the parabolic

equations could be derived from the mild-slope equation (Berkhoff, 1972).

The development of parabolic equation methods in water waves has grown rapidly in last five years. Weak reflection (Liu and Tsay, 1983), nonlinearity (Kirby and Dalrymple, 1983; Liu and Tsay, 1984), energy dissipation (Liu, 1986; Kirby and Dalrymple, 1986), and wave-current interactions (Booij, 1981; Liu, 1983; Dalrymple, Kirby and Hwang, 1984; Kirby, 1984) have all been included in the formulation. The parabolic equation methods have also been applied to several field problems (Dingemans, 1983; Liu et al., 1985). Many publications concerned with the parabolic approximation and related techniques have been summarized in a recent annotated bibliography (Liu, et al., 1986).

In this report, a brief review of the mild-slope equation and parabolic equations suitable for water waves is provided. Special emphasis is given to the physical meanings and justification of the parabolic approximation.

## 2. MILD-SLOPE EQUATION

In this section, we present a simple derivation of the mild-slope equation describing the propagation of small amplitude waves over a slowly varying topography. The derivation is very similar to that given by Smith and Sprinks (1975). The reader is also encouraged to consult the work of Berkhoff (1972) for a different derivation.

### 2.1 Derivation

We consider wave propagation over a gradually varying topography,  $z = -h(x, y)$ . The Cartesian coordinates  $(x, y, z)$  are fixed on the undisturbed water surface and the free surface displacement of small amplitude waves is described by  $z = \zeta(x, y, t)$ . Assuming that the fluid is inviscid and incompressible and the flow is irrotational, we introduce a velocity potential  $\phi(x, y, z, t)$ , so that the velocity vector,  $\vec{q}$ , is defined as the gradient of the potential; i.e.,  $\vec{q} = \nabla\phi$ .

Consider small amplitude monochromatic waves with a radian frequency  $\omega$ . The free surface boundary conditions can be linearized because of the smallness of the free surface displacement and the associated wave motions. Therefore, the time dependency in the surface elevation and the potential can be separated as follows:

$$\phi(x, y, x, t) = \phi(x, y, z)e^{-i\omega t} \quad (2.1)$$

$$\zeta(x, y, t) = \eta(x, y)e^{-i\omega t} \quad (2.2)$$

where only the real parts have physical meaning. The linearized boundary value problem for  $\phi$  can be written as

$$\frac{\partial^2 \phi}{\partial z^2} + \nabla^2 \phi = 0, \quad -h < z < 0 \quad (2.3)$$

$$\frac{\partial \phi}{\partial z} - \frac{\omega^2}{g} \phi = 0, \quad z = 0 \quad (2.4)$$

$$\frac{\partial \phi}{\partial z} = -\nabla h \cdot \nabla \phi, \quad z = -h \quad (2.5)$$

in which  $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$  is the two-dimensional gradient operator. Once the velocity potential is obtained, the free surface displacement can be found from the following kinematic free surface boundary condition:

$$\eta = \frac{i\omega}{g} \phi(x, y, 0) \quad (2.6)$$

For the special case of constant water depth, the right-hand side of (2.5) vanishes. The boundary value problem becomes a homogeneous one. Since the water depth is uniform, there is no wave refraction. To satisfy the boundary conditions (2.4), (2.5) and (2.6) we may rewrite the potential as

$$\phi = -\frac{ig}{\omega} \eta f \quad (2.7)$$

where

$$f = \frac{\cosh k(z+h)}{\cosh kh} \quad (2.8)$$

and  $k$  is the wave number which is the solution of the dispersion relation

$$\omega^2 = gk \tanh kh \quad (2.9)$$

From the Laplace equation (2.3), the free surface displacement must satisfy

the Helmholtz equation

$$\nabla^2 \eta + k^2 \eta = 0 \quad (2.10)$$

which is a two-dimensional reduced wave equation describing wave diffraction only, since  $k$  is a constant.

For slowly varying water depth, (2.7), (2.8) and (2.9) are still valid, with  $k$  and  $h$  referring to their local values. The governing equation for  $\eta$  should, however, be modified to include both refraction and diffraction. A simple method of obtaining the modified equation is presented herein.

The function  $f$  satisfies the following set of equations for slowly varying  $h$ :

$$\frac{\partial^2 f}{\partial z^2} - k^2 f = 0 \quad , \quad -h < z < 0 \quad (2.11)$$

$$\frac{\partial f}{\partial z} - \frac{\omega^2}{g} f = 0 \quad , \quad z = 0 \quad (2.12)$$

$$\frac{\partial f}{\partial z} = 0 \quad , \quad z = -h \quad (2.13)$$

provided that the dispersion relation, (2.9), is also satisfied. Considering (2.3) as an ordinary differential equation in  $z$ , and applying Green's second identity for  $\phi$  and  $f$  in the region  $-h < z < 0$ , we have

$$\int_{-h}^0 \left( \frac{\partial^2 \phi}{\partial z^2} f - \frac{\partial^2 f}{\partial z^2} \phi \right) dz = \left( \frac{\partial \phi}{\partial z} f - \frac{\partial f}{\partial z} \phi \right) \Big|_{z=0} - \left( \frac{\partial \phi}{\partial z} f - \frac{\partial f}{\partial z} \phi \right) \Big|_{z=-h}$$

Upon substituting (2.3) - (2.5) and (2.11) - (2.13) into the above equation, we obtain

$$\int_{-h}^0 (k^2 \phi f + f \nabla^2 \phi) dz = (f \nabla h + \nabla f) \Big|_{z=-h} \quad (2.14)$$

We now substitute (2.7) and (2.8) into (2.14) and note that

$$\nabla \phi = -\frac{ig}{\omega} (f \nabla n + n \frac{\partial f}{\partial h} \nabla h)$$

$$\nabla^2 \phi = -\frac{ig}{\omega} [f \nabla^2 n + 2n \frac{\partial f}{\partial h} \nabla n \cdot \nabla h + n \frac{\partial^2 f}{\partial h^2} (\nabla h)^2 + n \frac{\partial f}{\partial h} \nabla^2 h]$$

Equation (2.14) may be rewritten as

$$\begin{aligned} & \int_{-h}^0 \{f^2 \nabla^2 n + 2f \frac{\partial f}{\partial h} \nabla n \cdot \nabla h + n f \frac{\partial^2 f}{\partial h^2} (\nabla h)^2 \\ & + n f \frac{\partial f}{\partial h} \nabla^2 h + k^2 f^2 n\} dz = [f^2 \nabla h \cdot \nabla n + n f \frac{\partial f}{\partial h} (\nabla h)^2] \Big|_{z=-h} \end{aligned}$$

Upon using Leibniz' rule, the first two terms on the left of the above equation may be combined with the first term on the right, yielding

$$\begin{aligned} & \nabla \cdot (\int_{-h}^0 f^2 dz) \nabla n + k^2 (\int_{-h}^0 f^2 dz) n = f \frac{\partial f}{\partial h} \Big|_{z=-h} n (\nabla h)^2 \\ & - n \int_{-h}^0 [\frac{\partial^2 f}{\partial h^2} (\nabla h)^2 + \frac{\partial f}{\partial h} \nabla^2 h] f dz \end{aligned} \quad (2.15)$$

Because the water depth varies gradually within the distance of a typical wavelength, i.e.  $|\nabla h|/kh \ll 1$ , the right side of (2.15) is proportional to  $O(|\nabla h|/kh)^2$ , and can be neglected. Integrating  $f^2$  from  $z = -h$  to  $z = 0$ , we obtain

$$\nabla \cdot (CC_g \nabla n) + k^2 CC_g n = 0 \quad (2.16)$$

where

$$C = \frac{\omega}{k}, \quad C_g = \frac{1}{2} \frac{\omega}{k} (1 + \frac{2kh}{\sinh 2kh}) \quad (2.17)$$

are the phase velocity and the group velocity, respectively. Equation (2.16)

describes wave fields in which both refraction and diffraction are considered. Because of the assumption of gradually varying topography adopted in derivation, (2.16) is now called the mild-slope equation.

In the limiting case of arbitrary constant depth including the deep water case where  $kh \gg 1$ , (2.16) reduces to the Helmholtz equation (2.10). On the other hand, in the shallow water limit,  $kh \ll 1$ , (2.16) becomes

$$\nabla \cdot (h \nabla \eta) + \frac{\omega^2}{g} \eta = 0 \quad (2.18)$$

since  $C = C_g = \sqrt{gh}$  and  $\omega^2 = gk^2h$ . Equation (2.18) is the linear shallow water wave equation and can be derived directly from the continuity and momentum equations without using the mild-slope assumption. In other words, (2.18) is valid even for  $O(|\nabla h|/kh) = O(1)$ . Therefore, the mild-slope equation, (2.16), provides an interpolation for the whole range of wavelength.

Using the following transformation:

$$\eta = \xi / \sqrt{CC_g} \quad (2.19)$$

in (2.16), we obtain

$$\nabla^2 \xi + k_C^2 \xi = 0 \quad (2.20)$$

where the effective wave number  $k_C$  is defined by

$$k_C^2 = k^2 - \frac{\nabla^2(\sqrt{CC_g})}{\sqrt{CC_g}} \quad (2.21)$$

The second term on the right-hand side of (2.21) is of the order of  $O(|\nabla h|/kh)^2$ . Therefore, to be consistent with the order of magnitude of the accuracy of the mild-slope equation, the effective wave number can be approximated by the local wave number  $k$ . Thus

$$\nabla^2 \xi + k^2 \xi = 0 \quad (2.22)$$

which is the Helmholtz equation with a variable coefficient  $k$  being determined by the dispersion relation.

## 2.2 Relation Between Solutions of the Mild-Slope Equation and Ray Theory

In this section, we examine the relationship between solutions of the mild-slope equation and those of ray theory. To facilitate the analysis, we rewrite the free surface displacement as

$$\eta = A(x,y) e^{iS} \quad (2.23)$$

where both  $A$  and  $S$  are real functions. Substituting (2.23) into (2.16), we obtain

$$\begin{aligned} i[\nabla S \cdot (CC_g \nabla A) + \nabla \cdot (CC_g A \nabla S)] + \nabla \cdot (CC_g \nabla A) \\ - \nabla S \cdot (CC_g A \nabla S) + k^2 CC_g A = 0 \end{aligned}$$

Multiplying the above equation by  $A$  yields

$$\begin{aligned} A [CC_g \nabla^2 A + \nabla(CC_g) \cdot \nabla A - CC_g (|\nabla S|^2 - k^2) A] \\ + i \nabla \cdot (CC_g A^2 \nabla S) = 0 \end{aligned} \quad (2.24)$$

The real and imaginary parts of (2.24) should be zero. Thus

$$|\nabla S|^2 - k^2 = \frac{\nabla^2 A}{A} + \frac{\nabla(CC_g) \cdot \nabla A}{CC_g A} \quad (2.25)$$

and

$$\nabla \cdot (CC_g A^2 \nabla S) = 0 \quad (2.26)$$

Let us define

$$\vec{C}_g = C_g \frac{\nabla S}{|\nabla S|} \quad (2.27)$$

and substitute this into (2.26) to get

$$\nabla \cdot (\vec{C}_g A^2) + \left[ \frac{k}{|\nabla S|} \vec{C}_g \cdot \nabla \left( \frac{|\nabla S|}{k} \right) \right] A^2 = 0 \quad (2.28)$$

Equation (2.25) reduces to the well-known eikonal equation of ray theory, if the right-hand side terms are ignored. At the same time, (2.28) becomes the transport equation which requires the conservation of wave action  $A^2/\omega$  along a wave ray.

Therefore, by solving the mild-slope equation, the effects of diffraction, i.e., the gradient and the curvature of the amplitude, are taken into account. Wave action is no longer conserved along a "ray". Instead, (2.25) and (2.28) show that there is a gain or loss in wave action along a "ray" because of diffraction.

### 2.3 Energy Dissipation

In the previous sections the mild-slope equation, (2.16), and the ray theory were derived based on the assumption that no energy dissipation occurs during the wave propagation process. However, in most coastal engineering problems the energy dissipation effects, such as bottom friction and wave breaking, may become important. The mild-slope equation can be modified in a simple manner to accommodate these phenomena.

Let us consider  $W_r(x,y)$  as an energy dissipation function describing the rate of change of wave energy. The energy equation, (2.28), may be modified to be

$$\nabla \cdot (\vec{C}_g A^2) + \left[ \frac{k}{|\nabla S|} \vec{C}_g \cdot \nabla \left( \frac{|\nabla S|}{k} \right) \right] A^2 - \frac{k}{|\nabla S|} W_r A^2 = 0 \quad (2.29)$$

Since energy dissipation also affects the phase of the wave train, the equivalent eikonal equation, (2.25), should also be modified to be

$$|\nabla S|^2 - k^2 = \frac{\nabla^2 A}{A} + \frac{\nabla(CC_g)}{CC_g A} \cdot \nabla A + \frac{\omega W_i}{CC_g} \quad (2.30)$$

where  $W_i$  is a prescribed energy dissipation function influencing the phase function  $S$ . Combining (2.29) and (2.30) and introducing a complex energy dissipation function

$$W = W_r + i W_i \quad (2.31)$$

we find the corresponding mild-slope equation

$$\nabla \cdot (CC_g \nabla n) + k^2 CC_g \left( 1 - \frac{iW}{kC_g} \right) n = 0 \quad (2.32)$$

The energy dissipation function is determined according to different dissipative processes and could be a function of the free surface displacement,  $n$ , which makes (2.32) nonlinear (Dalrymple, Kirby and Hwang, 1984; Kirby and Dalrymple, 1986).

Using the same transformation as (2.19) for  $n$ , we obtain

$$\nabla^2 \xi + k_c^2 \xi = 0 \quad (2.20)$$

with

$$k_c^2 = k^2 - \frac{\nabla^2(\sqrt{CC_g})}{\sqrt{CC_g}} - \frac{ikW}{C_g} \quad (2.33)$$

If energy dissipation is ignored, (2.33) reduces to (2.21).

### 3. PARABOLIC APPROXIMATION AND PARABOLIC WAVE EQUATIONS

The mild-slope equations, (2.16) and (2.22), have been used to study various kinds of combined refraction and diffraction phenomena as boundary value problems (e.g., Smith and Sprinks, 1975; Skovgaard and Jonsson, 1980; Houston, 1981; Tsay and Liu, 1983). It is not, however, always convenient to treat every problem as a boundary value problem, especially if the boundary conditions are not well defined along the entire boundary. For instance, if a shoreline is part of the boundary, the boundary conditions for breaking waves along the shoreline are not certain. Furthermore, if the mild-slope equation is solved directly, the size of the finite elements (or grid cells) used in discretizing the computational domain must be a small fraction of the local wavelength. A huge number of elements (or nodes) may be necessary when the computational domain is large. An alternative computing method is needed to address regional (large scale) problems concerning wave propagation in coastal waters. The parabolic equation method, discussed in the next section, presents a possible alternative.

#### 3.1 Forward Scattering

The basic concept of the parabolic equation method is to convert the mild-slope equation to a set of approximate equations which describe a wave system propagating in a prescribed direction while still considering the energy flux in the transverse direction. Thus, both refraction and diffraction are included in this approximate formulation. There are different ways to obtain the parabolic equations, such as the matrix splitting method

(Corones, 1975; Radder, 1979) and the multi-scale perturbation method (Lozano and Liu, 1980). In this section, we present a simple approach, primarily due to Claerbout (1976) and Tappert (1977), to illustrate the method.

We assume that the primary wave propagation direction is in the x-direction. The modified free surface displacement  $\xi$ , (2.19), can be written as

$$\xi = \psi(x, y) e^{ik_0 x} \quad (3.1)$$

where  $k_0$  is a reference wave number and  $\psi$  is a slowly varying function of both  $x$  and  $y$  provided that the difference between  $k_0$  and  $k$  is small. Substituting (3.1) into (2.20), we obtain

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + 2ik_0 \frac{\partial \psi}{\partial x} + (k_c^2 - k_0^2) \psi = 0 \quad (3.2)$$

For the case of constant water depth without energy dissipation,  $k_0 = k_c = k$ , and (3.2) becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + 2ik \frac{\partial \psi}{\partial x} = 0 \quad (3.3)$$

The usual parabolic approximation assumes that the length scale of the amplitude variation in the x-direction (direction of wave propagation),  $L_x$ , is much longer than the length scale of the amplitude variation in the y-direction (transverse direction),  $L_y$ . Because  $\psi$  is a slowly varying function in  $x$  and  $y$ , the length scales  $L_x$  and  $L_y$  are much longer than the typical wavelength. In other words,

$$0 \left( \frac{L_y}{L_x} \right) = \epsilon \ll 1 ,$$

$$\begin{aligned} 0 \left( k \frac{L_x}{x} \right) &= 0 \left( \frac{L_x}{L_y} \right)^2 = 0 \left( \epsilon^{-2} \right) \\ 0 \left( k L_y \right) &= 0 \left( \frac{L_x}{L_y} \right) = 0 \left( \epsilon^{-1} \right) \end{aligned} \quad (3.4)$$

Because of these relationships among the length scales,  $\partial^2 \psi / \partial x^2$  is much smaller than  $\partial^2 \psi / \partial y^2$  and can be neglected. Therefore, (3.2) can be approximated as

$$\frac{\partial^2 \psi}{\partial y^2} + 2ik_0 \frac{\partial \psi}{\partial x} + (k_c^2 - k_0^2) \psi = 0 \quad (3.5)$$

The order of magnitude of the term dropped from (3.5) is of  $O(\epsilon^2)$ . If the third term in (3.5) is in the same order of magnitude as the other two terms, it is required from (3.4) that

$$\left( \frac{k_c}{k_0} \right)^2 - 1 = O(\epsilon^2) \quad (3.6)$$

If the water depth is a constant and energy dissipation is ignored, (3.3) can be simplified farther to be

$$\frac{\partial^2 \psi}{\partial y^2} + 2ik \frac{\partial \psi}{\partial x} = 0 \quad (3.7)$$

The above equation is in the same form as the heat equation if  $x$  is interpreted as time and the heat conductivity is taken to be imaginary. Equations (3.5) and (3.7) have the form of the linear Schrödinger equation.

To examine further the approximations involved in the parabolic wave equation, we can convert (3.5) or (3.7) back from the  $\psi$  variable to the  $\xi$  variable and compare them with the Helmholtz equation, (2.22); i.e.,

$$\nabla^2 \xi + k^2 \xi = 0 \quad (2.22)$$

For simplicity, we focus our discussion on the constant water depth without energy dissipation case. Substitution of (3.1) into (3.7) yields

$$\frac{\partial^2 \xi}{\partial y^2} + 2ik \frac{\partial \xi}{\partial x} + 2k^2 \xi = 0 \quad (3.8)$$

which is significantly different from the Helmholtz equation. The possible solutions for the above equation can be expressed as  $\exp[i(k_x x + k_y y)]$ , which represents a plane wave propagating in the  $\vec{k} = (k_x, k_y)$  direction. Substituting the exponential function solution into (3.8), we obtain an algebraic equation for  $k_x$  and  $k_y$ :

$$k_y^2 + 2k k_x - 2k^2 = 0 \quad (3.9)$$

Note that  $k_x$  is not equal to  $k$  unless  $k_y$  is zero. The difference between  $k$  and  $k_x$  is partly accounted for by the phase modification in  $\psi$ , (3.1).

Equation (3.9) is plotted in Figure 3.1, which shows a parabolic curve. Since only positive  $k_x$  values are allowable, the parabolic equation, (3.7), describes only forward scattering.

Substituting the exponential solution,  $\exp [i(k_x x + k_y y)]$  into the Helmholtz equation, we obtain the relation between  $k_y$ ,  $k_x$  and  $k$  as

$$k_x^2 + k_y^2 = k^2 \quad (3.10)$$

which represents a circle on the  $k_x$ ,  $k_y$  plane (Figure 3.1). It is clear from the figure that (3.9) is an approximation of (3.10) for small angle,  $\theta = \tan^{-1}(k_y/k_x)$ . Furthermore, (3.10) indicates that for a given  $k_y$  the sign of  $k_x$  could be either positive or negative. In other words, from the Helmholtz

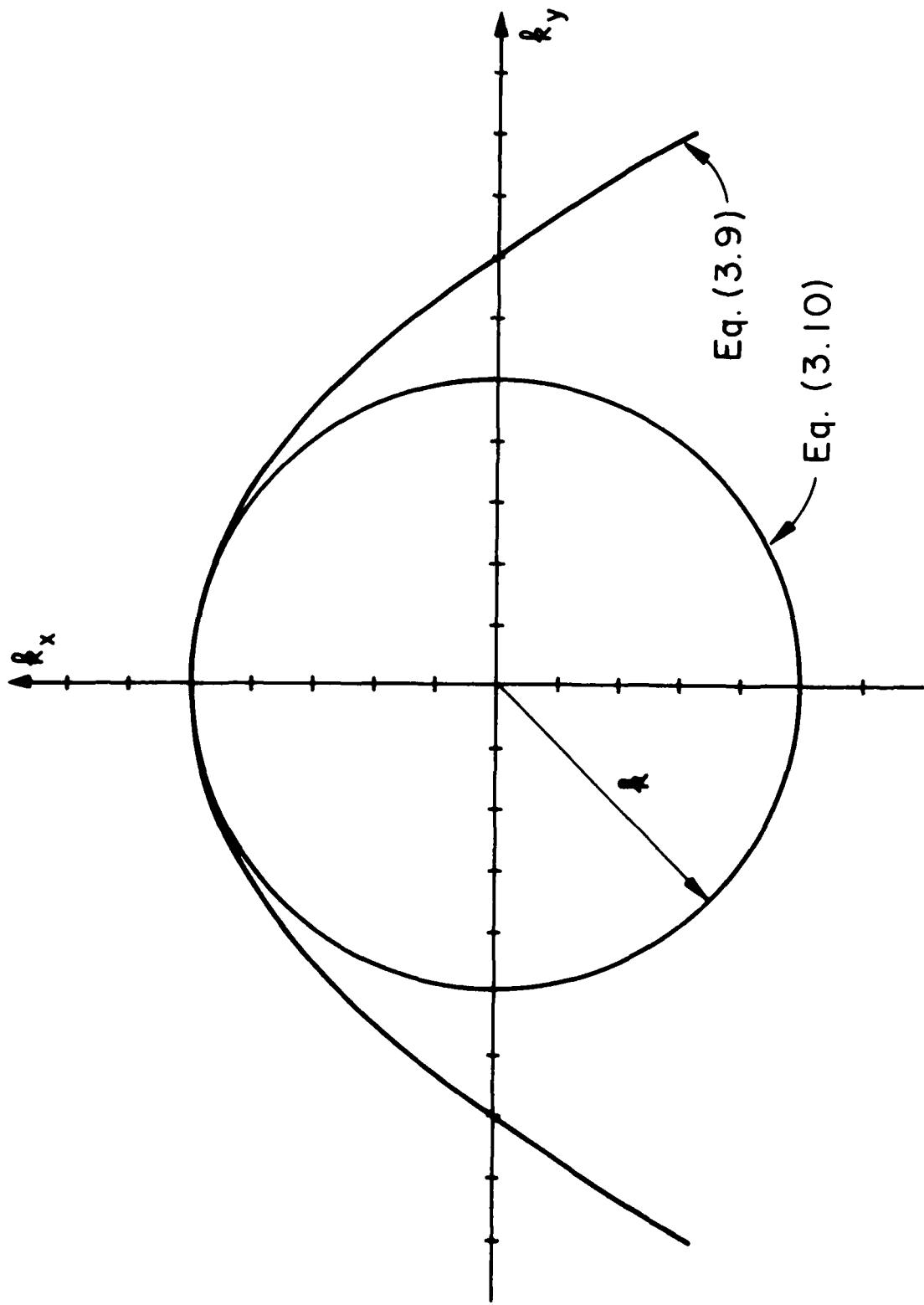


Figure 3.1 Wave number diagrams for the Helmholtz equation, (3.10), and the parabolic wave equation, (3.9).

equation waves may propagate in either the positive or negative  $x$ -direction with the same wave number  $k_x$ ; whereas, the parabolic approximation, (3.7), examines only forward scattering, with an additional small angle requirement.

The small angle requirement may be relaxed if the forward propagating wave equation has a semi-circle for its relation between  $k_x$  and  $k_y$ . The equation for a perfect semi-circle is given by

$$k_x = \sqrt{k^2 - k_y^2} \quad (3.11)$$

The corresponding wave equation is

$$i \frac{\partial \xi}{\partial x} + k Q \xi = 0 \quad (3.12)$$

where  $Q$  is a pseudo-differential operator given by

$$Q = (1 + \frac{1}{k^2} \frac{\partial^2}{\partial y^2})^{1/2} \quad (3.13)$$

(One can easily obtain (3.11) by substituting the exponential solution,  $\exp [i(k_x x + k_y y)]$  into (3.12)). As indicated in (3.4), for the parabolic approximation where  $O(kLy) \sim O(\epsilon^{-1})$ , (3.13) can be written as

$$Q = (1 + \epsilon^2)^{1/2} \quad (3.14)$$

For small  $\epsilon$ ,  $Q$  may be approximated using several different methods. For instance, using a binomial expansion, for small  $\epsilon$ ,  $Q$  may be written as

$$Q = (1 + \frac{1}{2k^2} \frac{\partial^2}{\partial y^2} + \frac{1}{8k^4} \frac{\partial^4}{\partial y^4} + \dots) \quad (3.15)$$

The corresponding wave equation becomes

$$i \frac{\partial \xi}{\partial x} + k (1 + \frac{1}{2k^2} \frac{\partial^2}{\partial y^2} + \frac{1}{8k^4} \frac{\partial^4}{\partial y^4} + \dots) \xi = 0 \quad (3.16)$$

By substituting the exponential solution,  $\exp [i(k_x x + k_y y)]$  into (3.16), the associated equation for  $k_x$  and  $k_y$  may be written as

$$k_x = k \left( 1 - \frac{k_y^2}{2k^2} - \frac{k_y^4}{8k^4} - \dots \right) \quad (3.17)$$

which is the binomial expansion of (3.11) for a small angle  $\tan^{-1}(k_y/k_x)$ . This expansion converges for all  $0 < k_y < k$ . Therefore, the parabolic approximation shown in (3.8) and (3.9) is the lowest order approximation of (3.16) and (3.17), being truncated at the second term of the series expansion. The effects of large angle diffraction may be studied if the higher order terms are kept in the analysis. In table 3.1, a comparison between the exact solution, (3.11), and the approximated solutions with two terms and three terms is given.

As an alternative to the binomial expansion, we can choose the rational expansion of  $Q$ , for small  $\epsilon$ ,

$$\begin{aligned} Q &= (1 + \epsilon^2)^{1/2} \approx \frac{1 + \frac{3}{4}\epsilon^2}{1 + \frac{1}{4}\epsilon^2} \\ &= (1 + \frac{3}{4k^2} \frac{\partial^2}{\partial y^2}) / (1 + \frac{1}{4k^2} \frac{\partial^2}{\partial y^2}) \end{aligned} \quad (3.18)$$

Substituting (3.18) into (3.12) yeilds the following wave equation

$$i \frac{\partial \xi}{\partial x} + \frac{i}{4k^2} \frac{\partial^3 \xi}{\partial x \partial y^2} + k \xi + \frac{3}{4k} \frac{\partial^2 \xi}{\partial y^2} = 0 \quad (3.19)$$

The corresponding equation for  $k_x$  and  $k_y$  can be written as

$$\frac{k_x}{k} = \frac{1 - \frac{3}{4} (\frac{k_y}{k})^2}{1 - \frac{1}{4} (\frac{k_y}{k})^2} \quad (3.20)$$

The accuracy of (3.20) compared with the exact solution is also shown in table 3.1. It is clear that the approximation given in (3.20) is the most accurate one for larger angles.

Substituting (3.1) into (3.16) and (3.19) with  $k=k_0$ , the higher-order parabolic approximations can be written in terms of  $\psi$  as

$$2ik \frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{4k^2} \frac{\partial^4 \psi}{\partial y^4} = 0 \quad (3.21)$$

$$2ik \frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial y^2} + \frac{i}{2k} \frac{\partial^3 \psi}{\partial x \partial y^2} = 0 \quad (3.22)$$

respectively. In fact, (3.21) can be obtained from (3.22) by using the first order approximation, (3.7), of  $\partial \psi / \partial x$  in the third term. If the water depth is a function of spatial coordinates or the energy dissipation is not negligible,  $k_c$  is no longer a constant. The higher-order parabolic wave equation can be written as follows:

$$2ik_0 \frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial y^2} + (k_c^2 - k_0^2) \psi + \frac{i}{2k_0} \frac{\partial^3 \psi}{\partial x \partial y^2} = 0 \quad (3.23)$$

Note that if the last term in the above equation is ignored, the lower-order parabolic approximation equation, (3.5), is recovered. Using a different approach as shown in the next section, the parabolic wave equations can be expressed in slightly different forms:

Table 3.1.  
 Comparison between the exact solution for  $k_x/k$  and  $\theta = \tan^{-1}(k_y/k_x)$ ,  
 using (3.11) and approximate solutions

$k_y/k$	Eq. (3.11)		Eq. (3.9)		Eq. (3.17)		Eq. (3.20)	
	$k_x/k$	$\theta$	$k_x/k$	$\theta$	$k_x/k$	$\theta$	$k_x/k$	$\theta$
0.00	1.0	0°	1.000	0°	1.000	0°	1.000	0°
0.10	0.995	5.739°	0.995	5.739°	0.995	5.739°	0.995	5.739°
0.30	0.954	17.458°	0.995	17.439°	0.954	17.458°	0.954	17.458°
0.50	0.866	30.000°	0.875	29.745°	0.867	29.969°	0.867	29.984°
0.70	0.714	44.429°	0.755	42.835°	0.725	43.999°	0.721	44.161°
0.80	0.600	53.130°	0.680	49.636°	0.629	51.833°	0.619	52.269°
0.85	0.527	58.211°	0.639	53.074°	0.574	55.988°	0.559	56.664°
0.90	0.436	64.160°	0.595	56.531°	0.513	60.317°	0.492	61.326°
0.95	0.312	71.802°	0.549	59.986°	0.469	64.807°	0.417	66.286°
1.00	0.000	90.0°	0.500	63.435°	0.375	69.444°	0.333	71.582°

$$2ik_c \frac{\partial \psi}{\partial x} + \left( \frac{3}{2} - \frac{k_0}{2k_c} \right) \frac{\partial^2 \psi}{\partial y^2} + [2k_c(k_c - k_0) + i \frac{\partial k_c}{\partial x}] \psi + \frac{i}{2k_c} \frac{\partial^3 \psi}{\partial x \partial y^2} = 0 \quad (3.24)$$

We remark here that since  $k_c - k_0 = O(\epsilon^2)$  from (3.6), (3.24) can be reduced to (3.23) by replacing  $k_c$  with  $k_0$  without losing accuracy.

The lower-order (small angle) parabolic wave equations, (3.5) and (3.7), are in the form of heat equation. Therefore, the wave propagation problem can now be viewed as an initial boundary value problem treating the direction of wave propagation as the time. The initial conditions in terms of wave amplitudes and phase functions must be prescribed along a straight line  $x = \text{constant}$ . The Crank-Nicolson finite difference scheme (e.g., Smith, 1978) can be used to discretize the parabolic wave equations. This scheme solves for the unknowns,  $\psi$ , along a line normal to the direction of wave propagation one step at a time. A detailed documentation on this numerical scheme can be found in Tsay and Liu (1986). Two lateral boundary conditions are required at the end points of the line. Two typical types of boundary usually occur in the combined refraction and diffraction problems: (1) a rigid boundary and (2) an open boundary. In the case of a rigid boundary,  $y = y_b(x)$ , the no-flux boundary condition requires

$$\vec{n} \cdot \nabla \eta = 0 \quad \text{on } y = y_b(x) \quad (3.25)$$

where  $\vec{n}$  is the unit normal along the solid boundary. Using (2.19) and (3.1) in (3.25) and employing the  $\vec{n} = \nabla[y - y_b(x)] / |\nabla[y - y_b(x)]|$ , we obtain

$$\frac{\partial \psi}{\partial y} - \frac{dy_b}{dx} \left[ \frac{\partial \psi}{\partial x} + ik\psi - \frac{\psi}{2} \frac{\partial CC_g}{\partial x} \right] - \frac{\psi}{2} \frac{\partial CC_g}{\partial y} = 0 , \quad y = y_b(x) \quad (3.26)$$

Further simplification may be made depending on the slope of the solid boundary. In the case of an open boundary, if the boundary is located sufficiently far away from the region of interest and the topography is assumed to be uniform in the  $y$ -direction near the open boundary, the approximate boundary condition can be written as

$$\frac{\partial \psi}{\partial y} = 0 \quad , \quad y = y_0 \quad (3.27)$$

The question concerning accurate and efficient numerical method for dealing with the open boundary condition remains open. The reader is encouraged to consult the work of Cohn and Jennings (1983), which discussed different methods for treating the silent open boundary.

Because  $\psi$  is a complex quantity, it may be written as

$$\psi = |\psi| e^{i\theta}, \quad \theta = \tan^{-1} \left( \frac{\text{Im } \psi}{\text{Re } \psi} \right)$$

The resulting wave number components in the  $x$ - and  $y$ -direction can be expressed as  $(k_0 + \partial\theta/\partial x, \partial\theta/\partial y)$ . The directions of wave propagation can be calculated accordingly.

### 3.2 Weak Reflection

In the previous section only the forward propagating wave field was considered. If wave reflection is important, several different approaches have been developed to derive a set of parabolic equations for both forward and backward scattering wave fields. The method presented here is similar to that given by Tappert (1977).

We first rewrite the Helmholtz equation in the following form:

$$\frac{\partial^2 \xi}{\partial x^2} + k_c^2 Q^2 \xi = 0 \quad (3.28)$$

where

$$Q^2 = \frac{1}{k_c^2} \frac{\partial^2}{\partial y^2} + 1 \quad (3.29)$$

in which  $k_c$  is given in (2.33). We remark here that in the case of a constant water depth and zero energy dissipation,  $k_c = k$ , and reflection can still be caused by the appearance of surface piercing structures. Equation (3.29) reduces to (3.13) in the limiting case.

We assume that the total wave field can be split into a transmitted (forward scattering) wave field and a reflected (backward scattering) wave field. Thus

$$\xi = \xi_+ + \xi_- \quad (3.30)$$

$$\frac{\partial \xi}{\partial x} = ik_c Q (\xi_+ - \xi_-) \quad (3.31)$$

From the equations above, we obtain

$$\xi_+ = \frac{1}{2} (\xi - \frac{i}{k_c Q} \frac{\partial \xi}{\partial x}) \quad (3.32)$$

$$\xi_- = \frac{1}{2} (\xi + \frac{i}{k_c Q} \frac{\partial \xi}{\partial x}) \quad (3.33)$$

Differentiating (3.32) and (3.33) with respect to  $x$  and using the governing equation (3.28) for  $\partial^2 \xi / \partial x^2$  and (3.31) for  $\partial \xi / \partial x$ , we obtain

$$\frac{\partial \xi_+}{\partial x} - ik_c Q \xi_+ + \frac{1}{2k_c Q} \frac{\partial k_c Q}{\partial x} \xi_+ = \frac{1}{2k_c Q} \frac{\partial k_c Q}{\partial x} \xi_- \quad (3.34)$$

$$\frac{\partial \xi_-}{\partial x} + ik_c Q \xi_- + \frac{1}{2k_c Q} \frac{\partial k_c Q}{\partial x} \xi_- = \frac{1}{2k_c Q} \frac{\partial k_c Q}{\partial x} \xi_+ \quad (3.35)$$

Using the rational expansion of the pseudo-differential operator,  $Q$ , (3.29) in (3.34) and (3.35), we can derive the following approximate equations:

$$\frac{\partial \xi_+}{\partial x} - ik_c \xi_+ + \frac{1}{4k_c^2} \frac{\partial^3 \xi_+}{\partial x \partial y^2} - \frac{3i}{4k_c} \frac{\partial^2 \xi_+}{\partial y^2} + \frac{1}{2k_c} \frac{\partial k_c}{\partial x} \xi_+ = \frac{1}{2k_c} \frac{\partial k_c}{\partial x} \xi_- \quad (3.36)$$

$$\frac{\partial \xi_-}{\partial x} + ik_c \xi_- + \frac{1}{4k_c^2} \frac{\partial^3 \xi_-}{\partial x \partial y^2} + \frac{3i}{4k_c} \frac{\partial^2 \xi_-}{\partial y^2} + \frac{1}{2k_c} \frac{\partial k_c}{\partial x} \xi_- = \frac{1}{2k_c} \frac{\partial k_c}{\partial x} \xi_+ \quad (3.37)$$

We remark that the left-hand side of the governing equation for the forward scattering wave field is exactly the same as that of (3.19) for a constant water depth and no energy dissipation ( $k_c = k$ ). Equations (3.36) and (3.37) denote a set of coupled equations for forward and backward scattering wave fields. In terms of the amplitude functions

$$\Psi_{\pm} = \xi_{\pm} e^{\mp ik_0 x} \quad (3.38)$$

equations (3.36) and (3.37) can be rewritten as

$$\begin{aligned} 2ik_c \frac{\partial \Psi_+}{\partial x} + [2k_c (k_c - k_0) + i \frac{\partial k_c}{\partial x}] \Psi_+ + \frac{i}{2k_c} \frac{\partial^3 \Psi_+}{\partial x \partial y^2} \\ + \left( \frac{3}{2} - \frac{k_0}{2k_c} \right) \frac{\partial^2 \Psi_+}{\partial y^2} = i \frac{2k_c}{\partial x} \Psi_- e^{-2ik_0 x} \\ 2ik_c \frac{\partial \Psi_-}{\partial x} - [2k_c (k_c - k_0) - i \frac{\partial k_c}{\partial x}] \Psi_- + \frac{i}{2k_c} \frac{\partial^3 \Psi_-}{\partial x \partial y^2} \end{aligned} \quad (3.39)$$

$$-\left(\frac{3}{2} - \frac{k_0}{2k_c}\right) \frac{\partial^2 \psi_-}{\partial y^2} = i \frac{\partial k_c}{\partial x} \psi_+ e^{2ik_0 x} \quad (3.40)$$

In principle, (3.39) and (3.40) can be solved simultaneously for  $\psi_+$  and  $\psi_-$ . Liu and Tsay (1983) developed an iterative numerical scheme for (3.39) and (3.40) with successive corrections on the reflected and transmitted wave fields. Their scheme can be briefly described as follows: First the reflection is entirely ignored; i.e.,  $\psi_-^0 = 0$ , and (3.39) becomes

$$\begin{aligned} & 2ik_c \frac{\partial \psi_+^0}{\partial x} + [2k_c (k_c - k_0) + i \frac{\partial k_c}{\partial x}] \psi_+^0 + \frac{i}{2k_c} \frac{\partial^3 \psi_+^0}{\partial x \partial y^2} \\ & + \left(\frac{3}{2} - \frac{k_0}{2k_c}\right) \frac{\partial^2 \psi_+^0}{\partial y^2} = 0 \end{aligned} \quad (3.41)$$

which is exactly the same as (3.24), and can be solved numerically. Equation (3.41) can be rewritten as  $P [\psi_+^0] = 0$  where  $P$  is the operator of (3.41). Once  $\psi_+^0$  is found, we can substitute it into the right-hand side of (3.40) to get the modified reflected wave field,  $\psi_-^1$ . Thus

$$P^* [\psi_-^1] = i \frac{\partial k_c}{\partial x} \psi_+^0 e^{2ik_0 x} \quad (3.42)$$

where  $P^*$  represents the operator on the left-hand side of (3.40). We can use the same numerical scheme as described in the previous section to solve (3.42) for  $\psi_-^1$ , which is in turn substituted into the right-hand side of (3.39) to find the improved  $\psi_+^1$ . The procedure should be repeated until the converged solutions are obtained.

#### 4. WEAKLY NONLINEAR WAVES

The parabolic approximation can also be applied to weakly nonlinear waves. Using multiple scale perturbation schemes, several researchers (Yue and Mei, 1980; Kirby and Dalrymple, 1983; and Liu and Tsay, 1984) have derived the cubic Schrödinger equation as the evolution equation describing the wave envelope of Stokes second-order waves. Here we present a simpler approach to obtain the same evolution equation.

First, we rewrite the Helmholtz equation, (2.22), in the following form:

$$\nabla^2 \xi + \frac{\omega^2 k}{g \tanh kh} \xi = 0 \quad (4.1)$$

The wave frequency in nonlinear wave theory depends not only on wave number,  $k$ , but also on the amplitude,  $a$ . The nonlinear dispersion relation has the general form

$$\omega^2 = \omega^2(k^2, a^2) \quad (4.2)$$

On the assumption that the wave amplitude is fairly small, we can approximate (4.2) in a Taylor's series as

$$\omega^2 = \omega_0^2(k^2) + [\frac{\partial \omega^2}{\partial (a^2)}]_0 a^2 \quad (4.3)$$

where  $\omega_0^2(k^2)$  represents the linear dispersion relation given in (2.9). For Stokes waves  $[\partial \omega^2 / \partial (a^2)]_0$  can be expressed as (Mei, 1983):

$$[\frac{\partial \omega^2}{\partial (a^2)}]_0 = \frac{gk^3 \tanh kh}{8 \sinh^4 kh} [8 + \cosh 4kh - 2 \tanh^2 kh] \quad (4.4)$$

We also point out that, from (2.19) and (3.1), the wave amplitude can be expressed in terms of  $\xi$  and  $\psi$  as

$$a = |\eta| = |\xi|/\sqrt{CC_g} = |\psi|/\sqrt{CC_g} \quad (4.5)$$

Following the analysis presented in section 3.1, we can readily obtain the parabolic equation for the nonlinear Stokes waves as

$$\frac{\partial^2 \psi}{\partial y^2} + 2ik_0 \frac{\partial \psi}{\partial x} + K |\psi|^2 \psi = 0 \quad (4.6)$$

where

$$K = \frac{k^4}{8C_g^2 \sinh^4 kh} [8 + \cosh 4kh - 2 \tanh^2 kh] \quad (4.7)$$

Equations (4.6) and (4.7) are identical to those derived by Yue and Mei (1980). The Crank-Nicolson scheme can still be used to solve the nonlinear cubic Schrödinger equation, (4.6). This equation has been used to examine the wave field in the neighborhood of caustics (e.g., Kirby and Dalrymple, 1983; Liu and Tsay, 1984).

## 5. CONCLUDING REMARKS

In this report, a brief review of the mild-slope equation and the parabolic wave equations suitable for water waves was given. The physical meanings and the limitations of various parabolic approximations were discussed. Parabolic wave equations including the effects of energy dissipations and non-linearity were derived.

Recently, Kirby (1986) presented a wave equation which includes the effects of the small-amplitude fast undulations in seafloor. An additional term was added to the mild-slope equation. The parabolic wave equations, similar to those presented in this report, can be derived.

The concept of the parabolic approximation presented in this report has also been extended to shallow-water wave problems (Liu, Yoon and Kirby, 1985). Instead of the mild-slope equation, the Boussinesq equation was used as the governing equation. Expressing the solution in a Fourier series, Liu, Yoon and Kirby derived a set of coupled parabolic wave equations for Fourier components.

In a companion report (Tsay and Liu, 1986), various numerical schemes for solving the parabolic equation are presented. The accuracy and the limitations of each scheme are discussed and compared.

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